A Computational Approach to Improving Bounds on the Hales–Jewett Numbers

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Abstract

We develop a SAT-based framework to study the Hales–Jewett numbers $HJ(4;2)$ and $HJ(3;3)$. Coloring the discrete cubes $[4]^n$ and $[3]^n$ is modeled as a Boolean satisfiability problem, where satisfiability indicates the existence of a coloring that admits no monochromatic combinatorial line. To find colorings that meet this constraint, we start from van der Waerden progression-free colorings and use algorithmic techniques to strategically narrow the SAT search space. Additionally, we introduce off-diagonal Hales–Jewett numbers, which formalize the avoidance of full monochromatic lines in one color while partially avoiding monochromatic lines in others.

Within this framework, we provide an improved lower bound on $HJ(3;3)$ and introduce machine-assisted proofs of the known lower bounds on $HJ(4;2)$ and $HJ(3;3)$. Additionally, we develop several heuristics to aid SAT solvers in navigating the Hales–Jewett search space. We also determine new exact values and bounds for the off-diagonal Hales–Jewett invariants.

1 Introduction

The Hales–Jewett theorem is a fundamental result in Ramsey theory, stating that for any fixed alphabet size k and number of colors r, every r-coloring of the discrete cube $[k]^n$ must admit a monochromatic combinatorial line once n is sufficiently large [10]. The associated Hales–Jewett numbers $HJ(k;r)$ indicate the smallest dimension in which this phenomenon becomes unavoidable. Computationally, determining $HJ(k;r)$ sits at the intersection of combinatorics and theoretical computer science. While the involved structures are highly regular, the search space expands so quickly that even small parameter values are beyond the reach of naive methods. Despite the theorem's fundamental importance, explicit values of $HJ(k;r)$ are known only in a few cases.

To convey how quickly the search space explodes, consider $HJ(4;2)$: the smallest n such that every 2-coloring of the discrete cube $[4]^n$ admits a monochromatic combinatorial line. For

$n = 3$, there are already $2^{4^3} = 2^{64} \approx 1.84 \times 10^{19}$ colorings. At $n = 4$, the count becomes $2^{4^4} = 2^{256} \approx 1.16 \times 10^{77}$, which is within a few orders of magnitude of the estimated number of atoms in the observable universe. The cube we study, $[4]^{12}$, has $2^{4^{12}} = 2^{16,777,216}$ colorings, a quantity with over five million digits.

Recent SAT advances have enabled progress in related Ramsey-type problems, especially in calculating van der Waerden numbers [9]. These developments suggest that a SAT-based approach to the Hales–Jewett problem has the potential to establish improved quantitative bounds.

1.1 Contributions of This Work

The primary contribution of this work is the development of a SAT-based framework for the Hales–Jewett problem, focusing on $HJ(4;2)$ and $HJ(3;3)$. We construct Boolean formulas that encode combinatorial lines (defined in Section 2) in the discrete cube $[k]^n$ and enforce the absence of monochromatic lines in all satisfying Boolean variable assignments. We lift arithmetic progression-free colorings derived from van der Waerden numbers into the Hales–Jewett cubes, and then use SAT solvers to validate them and search for extensions. This yields an improved lower bound

$$HJ(3;3) \geq 14,$$

and machine-verified proofs of known bounds

$$HJ(4;2) \geq 12 \quad \text{and} \quad HJ(3;3) \geq 13.$$

A second set of contributions concerns algorithmic improvements, namely heuristic search-space reductions and structural refinements of the SAT encoding. Several techniques are introduced and analyzed, including (i) parallelization through line decomposition, (ii) symmetry breaking via antipalindromic colorings, (iii) geometric line bounding, and (iv) high-incidence heuristics. These methods seek to decrease the size or effective complexity of the resulting SAT instances.

Finally, we present the off-diagonal Hales–Jewett numbers as an asymmetric Hales–Jewett invariant motivated by the avoidance of partial monochromatic combinatorial lines. We establish several new exact values and bounds on the off-diagonal analogs of $HJ(4;2)$ and $HJ(3;3)$.

1.2 Overview and Organization

Section 2 introduces the combinatorial structures of interest, including words, lines, and Hales–Jewett numbers. This section establishes the notational conventions used throughout this work.

Section 3 reviews known results on Hales–Jewett numbers, including exact values, upper bounds, and lower bounds. We provide proofs for $HJ(4;2) \geq 12$ and $HJ(3;3) \geq 13$, which serve as the logical foundation of our machine-assisted proofs in Section 5.

Section 4 offers a review of the Boolean satisfiability problem (SAT) and develops SAT encodings of discrete cubes that avoid monochromatic combinatorial lines. We outline how $HJ(4;2)$ and $HJ(3;3)$ are translated into SAT instances with a detailed discussion of Boolean formula construction and SAT solver implementation.

Section 5 explains how the proofs for lower bounds on $HJ(4;2)$ and $HJ(3;3)$ described in Section 3 are verified using SAT solvers. Additionally, we describe the process of extending the monochromatic combinatorial line-free coloring of $[3]^{12}$ to $[3]^{13}$, resulting in an improved lower bound of $HJ(3;3) \geq 14$.

Section 6 introduces four algorithmic methods aimed at improving SAT solver performance. Section 6.1 presents a parallelization technique that distributes solver workload across multiple processors. Section 6.2 explains how restrictive colorings, specifically antipalindromic colorings, reduce the Hales–Jewett search space. Section 6.3 discusses geometric lines and demonstrates how geometric Hales–Jewett numbers bound their combinatorial counterparts. Lastly, Section 6.4 outlines a heuristic designed to establish upper bounds on Hales–Jewett numbers by analyzing combinatorial lines through high-incidence points.

Section 7 introduces the off-diagonal Hales–Jewett numbers. We define these asymmetric invariants, explain how their Boolean encodings differ from those of their classical counterparts, report new computational results for these parameters, and conjecture a general value for $HJ(k, 2; 2)$.

Section 8 summarizes our work and suggests several natural directions for future research.

2 Preliminaries and Notation

In this section, we introduce the combinatorial structures relevant to our work and establish notation and conventions. All definitions are provided in their general form and naturally specialize to parameters of interest.

2.1 Words, Variable Words, and Combinatorial Lines

For a positive integer k, let

$$[k] = \{1, 2, \dots, k\}$$

denote the alphabet of k symbols. The set of all *words* of length n over this alphabet is

$$[k]^n = \{\mathbf{x} = (x_1, \dots, x_n) : x_i \in [k] \text{ for all } i \in [n]\}.$$

We view $[k]^n$ as a k-ary discrete n-dimensional cube.

A *variable word* is an element

$$w = (w_1, \dots, w_n) \in ([k] \cup \{\star\})^n$$

containing at least one occurrence of the special symbol $\star$. The set of *variable positions* is

$$S(w) := \{i \in [n] : w_i = \star\}, \;\; S(w) \neq \emptyset.$$

Substituting the letter $a \in [k]$ into every variable position produces the word

$$w(a) = (u_1, \dots, u_n), \qquad u_i = \begin{cases} a & \text{if } i \in S(w), \\ w_i & \text{if } i \notin S(w). \end{cases}$$

The *combinatorial line* generated by w is the set

$$L(w) := \{w(a) : a \in [k]\} \subseteq [k]^n.$$

Equivalently, a combinatorial line is determined by a choice of a nonempty set $S \subseteq [n]$ and a base word $\mathbf{x_0} \in [k]^n$. The line associated with $(S, \mathbf{x_0})$ is

$$L(S, \mathbf{x_0}) := \{\, \mathbf{x} \in [k]^n : x_i = (x_0)_i \text{ for } i \notin S, \text{ and } x_i \text{ is constant over } i \in S \,\}.$$

2.2 The Hales–Jewett Theorem and Hales–Jewett Numbers

The central result underlying this work is the Hales–Jewett theorem.

Theorem 2.1 (Hales–Jewett theorem [20])**.** *For every $k, r \in \mathbb{Z}_{>0}$, there exists a positive integer $HJ(k;r)$ such that for all $n \geq HJ(k;r)$, every r-coloring of the points in the discrete cube $[k]^n$ admits a monochromatic combinatorial line.*

Equivalently, for $n \geq HJ(k;r)$ and any coloring $C : [k]^n \to \mathcal{C}_r$, there exists a variable word $w \in ([k] \cup \{\star\})^n$ such that the associated line

$$L(w) = \{w(a) : a \in [k]\}$$

is monochromatic under C. We define the colors of an r-coloring by

$$\mathcal{C}_r := \{c_1, c_2, \ldots, c_r\}.$$

This motivates the following numerical invariant.

Definition 2.2 (Hales–Jewett numbers)**.** For $k, r \geq 1$, the *Hales–Jewett number $HJ(k;r)$* is the least integer n such that every r-coloring of $[k]^n$ admits a monochromatic combinatorial line.

In other words, $HJ(k;r)$ is the smallest dimension for which the Hales–Jewett phenomenon becomes unavoidable; $HJ(k;r)$ marks the threshold where colorings with no monochromatic combinatorial lines cease to exist.

2.3 Van der Waerden and Hales–Jewett Numbers

The Hales–Jewett theorem generalizes van der Waerden's classical result on arithmetic progressions, first proven in 1927 [23].

Definition 2.3 (van der Waerden numbers)**.** For integers $k, r \geq 1$, the *van der Waerden number $w(k;r)$* is the smallest integer N such that every r-coloring of $\{1, 2, \ldots, N\}$ admits a monochromatic k-term arithmetic progression.

The structure of a k-term arithmetic progression in $\{1, \ldots, N\}$ can be seen as a one-dimensional analog of a combinatorial line in $[k]^n$, and there is an important connection between these structures. Specifically, r-colorings of $\{1, \ldots, N\}$ that avoid monochromatic k-term progressions can be extended to r-colorings of $[k]^n$ (for suitable n) that avoid monochromatic lines.

In Section 3, we make this connection explicit and integrate it with known van der Waerden numbers to derive lower bounds on $HJ(4;2)$, $HJ(3;3)$, and $HJ(k;r)$ in general.

2.4 Geometric Lines

While our primary focus is on combinatorial lines, it is sometimes useful to consider a broader class of lines defined using two variable symbols, $\star$ and $\diamond$.

A *geometric variable word* is an element

$$w^* = (w_1^*, \ldots, w_n^*) \in ([k] \cup \{\star, \diamond\})^n$$

containing at least one occurrence of $\star$ or $\diamond$. We define the sets of variable positions

$$S_\star(w^*) := \{i \in [n] : w_i^* = \star\} \quad \text{and} \quad S_\diamond(w^*) := \{i \in [n] : w_i^* = \diamond\},$$

and require $S_\star(w^*) \cup S_\diamond(w^*) \neq \emptyset$.

For each $a \in [k]$, we obtain the word

$$w^*(a) = (u_1, \ldots, u_n) \in [k]^n$$

by substituting a and $(k + 1 - a)$ into the variable positions $\star$ and $\diamond$, respectively; specifically

$$u_i = \begin{cases} a & \text{if } i \in S_\star(w^*), \\ k + 1 - a & \text{if } i \in S_\diamond(w^*), \\ w_i^* & \text{if } i \notin S_\star(w^*) \cup S_\diamond(w^*). \end{cases}$$

The associated *geometric line* generated by w^* is the set

$$L^*(w^*) := \{\, w^*(a) : a \in [k] \,\} \subseteq [k]^n.$$

Every combinatorial line is a geometric line arising from a word w^* in which the symbol $\diamond$ does not appear, but not every geometric line is combinatorial. In particular, coordinates marked by $\diamond$ move in the reflected direction $(k + 1 - a)$ as a increases, so there exist geometric lines that cannot be captured by our combinatorial line construction.

3 Known Results and Bounds

This section reviews the current theoretical understanding of the Hales–Jewett numbers. We discuss known Hales–Jewett values and analyze upper bounds on $HJ(4; 2)$ and $HJ(3; 3)$, including both broad primitive recursive estimates and more specific results.

Then, we review relevant van der Waerden numbers and present a lifting construction that converts arithmetic progression-free colorings of the positive integers into colorings of discrete cubes that avoid monochromatic combinatorial lines. Finally, we leverage this connection to prove lower bounds $HJ(4; 2) \geq 12$ and $HJ(3; 3) \geq 13$.

3.1 Exact Values

Only a limited number of values for $HJ(k;r)$ are currently known. The binary case is completely determined: for all $r \in \mathbb{Z}_{>0}$,

$$HJ(2;r) = r.$$

The equality is based on a brief combinatorial argument: it is possible to construct an r-coloring of $[2]^{r-1}$ that admits no monochromatic line, while it can be shown that every r-coloring of $[2]^r$ necessarily admits such a line (see, for example, [20]).

Beyond this case, only one nontrivial value has been computed:

$$HJ(3;2) = 4.$$

In this case, an explicit monochromatic line-free 2-coloring of $[3]^3$ establishes the lower bound $HJ(3;2) \geq 4$, while Hindman and Tressler [12] demonstrate that every 2-coloring of $[3]^4$ admits a monochromatic line using a structural forcing argument and an FPGA-assisted exhaustive computation. No other exact value of $HJ(k;r)$ is currently known.

3.2 General Upper Bounds

The earliest upper bounds for the Hales–Jewett numbers originate from Shelah's 1988 proof demonstrating that $HJ(k;r)$ is primitive recursive [21]. His argument offers explicit tower-type bounds dependent on k and r, including

$$HJ(4;2) \leq \underbrace{2^{2^{2^{2^{2^{2^{2^{2}}}}}}}}_{8\ \text{twos}} \approx 10^{10^{10^{10^{19848}}}}.$$

Shelah's bound for $HJ(3;3)$ is so large and intricate in its construction that it offers no practical numerical guidance. These values lie far beyond any feasible scale for explicit computation or direct search, rendering the bounds unusable for computational purposes.

For the specific case of $HJ(4;2)$, Lavrov [15] achieved a significant improvement by using the Lovász Local Lemma to obtain the much smaller bound

$$HJ(4;2) \leq 10^{11}.$$

In a similar vein, Conlon [6] proved that there exists a constant c such that

$$HJ(3;3) \leq 2^{2^{3c}}.$$

Although these estimates are still considered to be much larger than the actual values, they are significant improvements over Shelah's general bound.

3.3 Van der Waerden–based Lower Bounds

We demonstrate how arithmetic progression-free colorings of the positive integers derived from van der Waerden numbers lead to monochromatic line-free colorings of $[4]^{11}$ and $[3]^{12}$, proving lower bounds $HJ(4;2) \geq 12$ and $HJ(3;3) \geq 13$.

3.3.1 Relevant van der Waerden Numbers

The values relevant to $HJ(4;2)$ and $HJ(3;3)$, respectively, are

$$w(4;2) = 35, \qquad w(3;3) = 27,$$

which have been established through a combination of combinatorial arguments and computer search [5]. In particular, there exist r-colorings of $\{1, \dots, w(k;r)-1\}$ that avoid monochromatic k-term arithmetic progressions, and these colorings can be written down explicitly.

3.3.2 The Lifting Map

For integers $k, n \geq 1$ we define

$$f_{k,n} : [k]^n \to \mathbb{Z}_{>0}, \qquad f_{k,n}(\mathbf{x}) := 1 + \sum_{i=1}^{n} (x_i - 1).$$

Thus, $f_{k,n}(\mathbf{x})$ is the sum of the coordinates of $\mathbf{x}$, up to an additive constant. Many points in $[k]^n$ have the same image under $f_{k,n}$, but the crucial property for our purposes is the behavior of $f_{k,n}$ on combinatorial lines.

Let w be a variable word with variable position set $S(w)$ of size t, and let

$$L(w) = \{w(1), w(2), \dots, w(k)\}$$

be the associated combinatorial line. A direct calculation shows that

$$f_{k,n}(w(a+1)) - f_{k,n}(w(a)) = t \qquad \text{for all } a \in \{1, \dots, k-1\}.$$

In particular, the sequence

$$f_{k,n}(w(1)),\ f_{k,n}(w(2)),\ \dots,\ f_{k,n}(w(k))$$

is a k-term arithmetic progression with common difference t.

3.3.3 Progression-free Colorings Yield Line-free Colorings

Let $C_{VW,[N]} : \{1, \dots, N\} \to \mathcal{C}_r$ be an r-coloring of $[N]$ that avoids monochromatic k-term arithmetic progressions. The lifting map induces a coloring

$$C_{HJ(k;r),n} : [k]^n \to \mathcal{C}_r, \qquad C_{HJ(k;r),n}(\mathbf{x}) := C_{VW,[N]}(f_{k,n}(\mathbf{x})).$$

Then $L(w) \subseteq [k]^n$ with associated sequence

$$f_{k,n}(w(1)),\ f_{k,n}(w(2)),\ \dots,\ f_{k,n}(w(k))$$

forms a k-term arithmetic progression. Since $[N]$ contains no monochromatic k-term progressions, the values $C_{VW,[N]}(f_{k,n}(w(a)))$ cannot all coincide. Therefore $C_{HJ(k;r),n}$ is not constant on $L(w)$ and consequently it admits no monochromatic combinatorial line.

We now apply this method to construct explicit colorings of $[4]^{11}$ and $[3]^{12}$. These constructions yield the classical van der Waerden–based lower bounds on $HJ(4;2)$ and $HJ(3;3)$.

3.4 A Lower Bound for $HJ(4;2)$

We begin with the two-color, four-letter alphabet case.

Theorem 3.1. *$HJ(4;2) \geq 12$.*

Proof. The van der Waerden number $w(4;2) = 35$ guarantees the existence of a 2-coloring

$$C_{VW,[34]} : \{1, \ldots, 34\} \to \{c_1, c_2\}$$

that avoids monochromatic 4-term arithmetic progressions.

For $\mathbf{x} = (x_1, \ldots, x_{11}) \in [4]^{11}$, each coordinate satisfies $1 \leq x_i \leq 4$, so $0 \leq x_i - 1 \leq 3$ and therefore

$$1 \ \leq \ f_{4,11}(\mathbf{x}) = 1 + \sum_{i=1}^{11}(x_i - 1) \ \leq \ 1 + 11 \cdot 3 = 34.$$

Thus, $f_{4,11}(\mathbf{x}) \in \{1, \ldots, 34\}$ for all $\mathbf{x} \in [4]^{11}$, and the lifted coloring

$$C_{HJ(4;2),11} : [4]^{11} \to \{c_1, c_2\}, \qquad C_{HJ(4;2),11}(\mathbf{x}) := C_{VW,[34]}\big(f_{4,11}(\mathbf{x})\big)$$

is well-defined.

Let $L(w)$ be a combinatorial line in $[4]^{11}$. The sequence

$$f_{4,11}(w(1)),\ f_{4,11}(w(2)),\ f_{4,11}(w(3)),\ f_{4,11}(w(4))$$

forms a 4-term arithmetic progression.

Since $C_{VW,[34]}$ admits no 4-term monochromatic progression on [34], the values $C_{VW,[34]}(f_{4,11}(w(a)))$ cannot all be equal. Hence, $C_{HJ(4;2),11}$ is not constant on $L(w)$, and $L(w)$ is not monochromatic.

$C_{HJ(4;2),11}$ is therefore a 2-coloring of $[4]^{11}$ that admits no monochromatic combinatorial line, and it follows that $HJ(4;2) \geq 12$. ☐

3.5 A Lower Bound for $HJ(3;3)$

We now address the three-color, three-letter alphabet case.

Theorem 3.2. *$HJ(3;3) \geq 13$.*

Proof. The van der Waerden number $w(3;3) = 27$ guarantees the existence of a 3-coloring

$$C_{VW,[26]} : \{1, \ldots, 26\} \to \{c_1, c_2, c_3\}$$

that avoids monochromatic 3-term arithmetic progressions.

For $\mathbf{x} = (x_1, \ldots, x_{12}) \in [3]^{12}$, we have $1 \leq x_i \leq 3$, so $0 \leq x_i - 1 \leq 2$, and

$$1 \ \leq \ f_{3,12}(\mathbf{x}) = 1 + \sum_{i=1}^{12}(x_i - 1) \ \leq \ 1 + 12 \cdot 2 = 25.$$

Thus, $f_{3,12}(\mathbf{x}) \in \{1, \dots, 25\} \subseteq \{1, \dots, 26\}$ for all $\mathbf{x} \in [3]^{12}$, and the lifted coloring

$$C_{HJ(3;3),12} : [3]^{12} \to \{c_1, c_2, c_3\}, \qquad C_{HJ(3;3),12}(\mathbf{x}) := C_{VW,[26]}(f_{3,12}(\mathbf{x}))$$

is well-defined.

Consider $L(w) \subseteq [3]^{12}$ and its associated sequence

$$f_{3,12}(w(1)),\ f_{3,12}(w(2)),\ f_{3,12}(w(3)),$$

which forms a 3-term arithmetic progression. Since $C_{VW,[26]}$ admits no monochromatic 3-term progression on [26], the colors assigned to $w(a)$ by $C_{HJ(3;3),12}$ cannot all be equal.

Thus, $C_{HJ(3;3),12}$ is a 3-coloring of $[3]^{12}$ that avoids monochromatic combinatorial lines, and it follows that $HJ(3;3) \geq 13$. □

3.6 A General van der Waerden Lower Bound

These constructions can be generalized to arbitrary Hales–Jewett parameters (k, r). In particular, we may extend monochromatic k-term progression-free r-colorings of $[N]$ to monochromatic line-free colorings of $[k]^n$ via $C_{HJ(k;r),n}$, provided n is not too large.

Proposition 3.3. *Let $k \geq 2$ and $r \geq 1$, and let*

$$C_{VW,[N]} : \{1, \dots, N\} \to \mathcal{C}_r$$

be an r-coloring that avoids monochromatic k-term arithmetic progressions. Let $n \in \mathbb{Z}_{>0}$ such that

$$1 + n(k-1) \ \leq\ N.$$

Then there exists an r-coloring of $[k]^n$ that admits no monochromatic combinatorial line. In particular, $HJ(k;r) \geq n+1$.

Proof. Define the lifted coloring

$$C_{HJ(k;r),n} : [k]^n \to \mathcal{C}_r, \qquad C_{HJ(k;r),n}(\mathbf{x}) := C_{VW,[N]}\Big(f_{k,n}(\mathbf{x})\Big),$$

where

$$f_{k,n}(\mathbf{x}) := 1 + \sum_{i=1}^{n} (x_i - 1).$$

Let $\mathbf{x} \in [k]^n$. Then $0 \leq x_i - 1 \leq k-1$, and

$$1 \ \leq\ f_{k,n}(\mathbf{x}) \ \leq\ 1 + n(k-1) \ \leq\ N,$$

so $C_{HJ(k;r),n}$ is well-defined.

Let $L(w) \subseteq [k]^n$ be a combinatorial line. Then the associated sequence

$$f_{k,n}(w(1)),\, f_{k,n}(w(2)),\, \dots,\, f_{k,n}(w(k))$$

forms a k-term arithmetic progression. Since $C_{VW,[N]}$ avoids monochromatic k-term progressions on $[N]$, the colors assigned to $w(a)$ by $C_{HJ(k;r),n}$ cannot all coincide.

Thus, $C_{HJ(k;r),n}$ is not constant on $L(w)$. It follows that $C_{HJ(k;r),n}$ admits no monochromatic combinatorial line on $[k]^n$, and $HJ(k;r) \geq n+1$. $\square$

A direct consequence is a general lower bound for Hales–Jewett numbers in terms of van der Waerden numbers.

Corollary 3.4. *Let $k \geq 2$ and $r \geq 1$, and let $w(k;r)$ denote the corresponding van der Waerden number. Then*

$$HJ(k;r) \;\geq\; \left\lfloor \frac{w(k;r)-2}{k-1} \right\rfloor + 1.$$

Proof. There exists an r-coloring of $\{1,\ldots,w(k;r)-1\}$ that admits no monochromatic k-term progression.

Let $N := w(k;r)-1$ and choose

$$n := \left\lfloor \frac{N-1}{k-1} \right\rfloor = \left\lfloor \frac{w(k;r)-2}{k-1} \right\rfloor.$$

Then

$$1 + n(k-1) \;\leq\; N,$$

so Proposition 3.3 produces an r-coloring of $[k]^n$ that admits no monochromatic combinatorial line. Hence $HJ(k;r) \geq n+1$. $\square$

The bounds $HJ(4;2) \geq 12$ and $HJ(3;3) \geq 13$ obtained in Theorems 3.1 and 3.2 are exactly the corresponding cases of Corollary 3.4 using the values $w(4;2) = 35$ and $w(3;3) = 27$, respectively.

Discussion. If a bound on $w(k;r)$ is replaced by a larger value $w'(k;r) > w(k;r)$, then the resulting lower bound on $HJ(k;r)$ increases by at most

$$\left\lfloor \frac{w'(k;r)-2}{k-1} \right\rfloor - \left\lfloor \frac{w(k;r)-2}{k-1} \right\rfloor.$$

Recent work shows that $w(k;2) > 2^k$ [14], which in turn yields the approximate lower bound $HJ(k;2) \gtrsim \frac{2^k}{k}$. This illustrates that even a significant increase in a lower bound on $w(k;r)$ results in only a modest improvement in the corresponding Hales–Jewett lower bound. Since exact values of $w(k;r)$ are known for only a limited range of parameters, van der Waerden numbers alone are therefore unlikely to provide sharp bounds for $HJ(k;r)$.

To achieve stronger results, it is natural to seek methods that operate directly within the high-dimensional cube $[k]^n$, rather than on one-dimensional projections. In the next section, we pursue this approach by framing the Hales–Jewett problem as a Boolean satisfiability problem and using modern SAT solvers to find monochromatic combinatorial line-free colorings.

4 SAT Formulation of the Hales–Jewett Problem

In this section, we explain how to convert the search for monochromatic combinatorial line-free colorings of $[k]^n$ into an instance of the Boolean satisfiability problem (SAT).

We begin by reviewing core ideas from propositional logic and computer science: Boolean formulas and conjunctive normal form (CNF), the SAT problem they induce, and the main principles behind modern SAT solvers. We present a general SAT-encoding framework for Hales–Jewett colorings, then specialize it to $HJ(4;2)$ and $HJ(3;3)$. For these cases, we detail the point-to-Boolean variable mappings, the construction of SAT clauses, and the resulting instance sizes.

4.1 Boolean Variables, CNF, and SAT

A Boolean function is a map

$$f : \{0,1\}^n \to \{0,1\}.$$

Given Boolean variables $b_1, \ldots, b_n$, any assignment of truth values

$$(b_1, \ldots, b_n) \in \{0,1\}^n$$

produces an output $f(b_1, \ldots, b_n) \in \{0,1\}$.

Boolean functions are represented by formulas built from variables using the logical connectives conjunction ($\wedge$), disjunction ($\vee$), and negation ($\neg$). Conjunction plays a central role: a formula

$$\Phi = \Phi_1 \wedge \Phi_2 \wedge \cdots \wedge \Phi_m$$

is true if and only if every subformula Φ_i is true.

A formula is in *conjunctive normal form* (CNF) if it is a conjunction of *clauses*, where each clause is a disjunction of *literals* (variables or their negations). For example,

$$(b_1 \vee \neg b_2) \;\wedge\; (\neg b_3 \vee b_4 \vee \neg b_5)$$

is a CNF formula with two clauses.

The *Boolean satisfiability problem* (SAT) asks whether a CNF formula has a satisfying assignment. Equivalently, viewing Φ as a Boolean function $\Phi : \{0,1\}^n \to \{0,1\}$, SAT asks whether there exists an assignment

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n$$

such that

$$\Phi(\alpha_1, \ldots, \alpha_n) = 1.$$

If such an assignment exists, Φ is *satisfiable*; otherwise it is *unsatisfiable.* Although SAT is NP-complete [7], modern solvers routinely handle instances with millions of variables and clauses in practice.

Modern SAT Solvers and Computational Tools. Modern SAT solvers rely on conflict-driven clause learning (CDCL). The solver gradually constructs a partial assignment, uses unit propagation to determine implied variable values, and backtracks immediately when a contradiction arises. By incorporating advanced branching heuristics and carefully designed data structures, these solvers can explore large combinatorial search spaces with greater efficiency than brute-force approaches (see [16], [17], and [4] for a detailed discussion).

We use PySAT [13] to generate CNF formulas and export them to standard DIMACS format [8]. We solve the resulting instances with CryptoMiniSat [22], a modern solver that supports multi-core processing. All large-scale experiments are conducted on Colgate University's High-Performance Computing resources [1].

4.2 Encoding the Hales–Jewett Problem as SAT

Let $k, r, n \in \mathbb{Z}_{>0}$. We seek to determine whether there exists a coloring

$$C_{HJ} : [k]^n \to \mathcal{C}_r$$

such that for every combinatorial line $L \subseteq [k]^n$, the restriction $C_{HJ}|_L$ is not constant. We encode this existence statement into a Boolean formula $\Phi_{[k]^n}$, constructed so that $\Phi_{[k]^n}$ is satisfiable if and only if such a coloring C_{HJ} exists.

If $\Phi_{[k]^n}$ is satisfiable, then the resulting Boolean assignment is an r-coloring of $[k]^n$ that avoids monochromatic lines and hence $HJ(k; r) \geq n + 1$.

If $\Phi_{[k]^n}$ is unsatisfiable, then every r-coloring of $[k]^n$ admits at least one monochromatic combinatorial line, and $HJ(k; r) \leq n$.

We increase n until $\Phi_{[k]^n}$ is unsatisfiable. Since $\Phi_{[k]^{n-1}}$ is satisfiable, we have $HJ(k; r) \geq n$, while unsatisfiability of $\Phi_{[k]^n}$ gives $HJ(k; r) \leq n$. Hence, $HJ(k; r) = n$.

Enumerating Combinatorial Lines. A combinatorial line in $[k]^n$ is specified by choosing a nonempty variable position set S with $|S| = t \leq n$ and assigning letters from $[k]$ to the remaining $n - t$ coordinates. There are $\binom{n}{t}$ possible choices for the variable positions and k^{n-t} choices for the fixed symbols. Therefore, there are

$$\sum_{t=1}^{n} \binom{n}{t} k^{n-t}$$

combinatorial lines in the discrete cube $[k]^n$. By the binomial theorem,

$$\sum_{t=1}^{n} \binom{n}{t} k^{n-t} = (k + 1)^n - k^n.$$

Indexing the Cube. To express the cube $[k]^n$ as a SAT instance, every point in $[k]^n$ must be assigned a unique index. We use the standard base-k bijection

$$\mathrm{idx}_{(k,n)} : [k]^n \to \{1, \dots, k^n\}, \qquad \mathrm{idx}_{(k,n)}(\mathbf{x}) = 1 + \sum_{i=1}^{n} k^{n-i}(x_i - 1).$$

Equivalently, $\mathrm{idx}_{(k,n)}(\mathbf{x}) - 1$ is the base-k integer whose digits are $(x_1 - 1, \dots, x_n - 1)$. Each point of $[k]^n$ corresponds to a unique Boolean variable (in the case $r = 2$) or to a fixed sequence of Boolean variables (in the case $r > 2$) determined by $\mathrm{idx}_{(k,n)}$. Monochromatic combinatorial line constraints can then be written directly in terms of the literals associated with the points in the cube.

4.3 Encoding $HJ(4; 2)$

4.3.1 Variables and Indexing

In the 2-color case, each point $\mathbf{x} \in [4]^n$ is represented by a single Boolean variable b_i:

$$b_i = \begin{cases} 1, & \text{if } \mathbf{x} \text{ is } c_1\text{-colored}, \\ 0, & \text{if } \mathbf{x} \text{ is } c_2\text{-colored}. \end{cases}$$

To assign points to Boolean variable values, we use the base-4 indexing function

$$\mathrm{idx}_{(4,n)} : [4]^n \to \{1, \dots, 4^n\}, \qquad \mathrm{idx}_{(4,n)}(\mathbf{x}) = 1 + \sum_{i=1}^{n} 4^{n-i}(x_i - 1).$$

Each $\mathbf{x}$ corresponds to exactly one literal $b_{\mathrm{idx}_{(4,n)}(\mathbf{x})}$.

4.3.2 Problem Size

The cube $[4]^n$ contains 4^n points and therefore admits 2^{4^n} possible 2-colorings. By the general enumeration result in Section 4.2, there are

$$(4 + 1)^n - 4^n = 5^n - 4^n$$

combinatorial lines in $[4]^n$.

For $n = 12$, the cube $[4]^{12}$ has

$$4^{12} = 16{,}777{,}216 \text{ points}, \qquad 5^{12} - 4^{12} = 227{,}363{,}409 \text{ combinatorial lines},$$

and admits $2^{4^{12}} = 2^{16{,}777{,}216}$ possible colorings.

4.3.3 Line Clauses

Let $w \in ([4] \cup \{\star\})^n$ be a variable word, and let

$$L(w) = \{w(1), w(2), w(3), w(4)\} \subseteq [4]^n$$

be its associated combinatorial line. Using the indexing map $\mathrm{idx}_{(4,n)} : [4]^n \to \{1, \ldots, 4^n\}$, set

$$i_a := \mathrm{idx}_{(4,n)}\Big(w(a)\Big), \qquad a \in \{1, 2, 3, 4\},$$

so that point $w(a)$ corresponds to literal b_{i_a}. To forbid $L(w)$ from being monochromatic, we add the two clauses

$$(\neg b_{i_1} \vee \neg b_{i_2} \vee \neg b_{i_3} \vee \neg b_{i_4}) \qquad \text{and} \qquad (b_{i_1} \vee b_{i_2} \vee b_{i_3} \vee b_{i_4})$$

to the SAT CNF formula. The first clause rules out the all-c_1 assignment on $L(w)$ (all $b_{i_a} = 1$), while the second rules out the all-c_2 assignment (all $b_{i_a} = 0$). Together, they enforce that no combinatorial line in $[4]^n$ is monochromatic.

4.3.4 Final CNF Instance

Let $\mathcal{L}$ denote the set of combinatorial lines in $[4]^n$. The full CNF formula is

$$\Phi_{[4]^n} = \bigwedge_{L \in \mathcal{L}} \Big[(\neg b_{i_1} \vee \neg b_{i_2} \vee \neg b_{i_3} \vee \neg b_{i_4}) \wedge (b_{i_1} \vee b_{i_2} \vee b_{i_3} \vee b_{i_4})\Big],$$

where $(i_1, \ldots, i_4)$ are the Boolean indices associated with the points of L as above. Satisfiability of $\Phi_{[4]^n}$ is equivalent to the existence of a 2-coloring of $[4]^n$ that avoids monochromatic combinatorial lines.

4.4 Encoding $HJ(3;3)$

4.4.1 Variables and Indexing

In the 3-color case, each point $\mathbf{x} \in [3]^n$ is represented by three Boolean variables $(b_{i,c_1}, b_{i,c_2}, b_{i,c_3})$:

$$b_{i,c_1} = \begin{cases} 1, & \text{if } \mathbf{x} \text{ is } c_1\text{-colored}, \\ 0, & \text{otherwise}, \end{cases} \qquad b_{i,c_2} = \begin{cases} 1, & \text{if } \mathbf{x} \text{ is } c_2\text{-colored}, \\ 0, & \text{otherwise}, \end{cases} \qquad b_{i,c_3} = \begin{cases} 1, & \text{if } \mathbf{x} \text{ is } c_3\text{-colored}, \\ 0, & \text{otherwise}. \end{cases}$$

At least one of $(b_{i,c_1}, b_{i,c_2}, b_{i,c_3})$ must be true for each point. We do not require these variables to be mutually exclusive: more than one of $b_{i,c_1}, b_{i,c_2}, b_{i,c_3}$ may be true. This yields a relaxed (multi-color) encoding.

Given a satisfying multi-color assignment, we can select one true color at each point to obtain a genuine 3-coloring that avoids monochromatic lines.

We use the base-3 indexing map

$$\mathrm{idx}_{(3,n)} : [3]^n \to \{1, 2, \ldots, 3^n\}, \qquad \mathrm{idx}_{(3,n)}(\mathbf{x}) = 1 + \sum_{i=1}^{n} 3^{\,n-i}(x_i - 1).$$

Each cube point then corresponds to a triple of Boolean variables

$$(b_{\mathrm{idx}_{(3,n)}(\mathbf{x}),c_1}, b_{\mathrm{idx}_{(3,n)}(\mathbf{x}),c_2}, b_{\mathrm{idx}_{(3,n)}(\mathbf{x}),c_3}).$$

4.4.2 Problem Size

The discrete cube $[3]^n$ contains 3^n points and therefore admits 3^{3^n} possible 3-colorings. By the general enumeration result in Section 4.2, there are

$$(3+1)^n - 3^n = 4^n - 3^n$$

combinatorial lines in $[3]^n$.

For $n = 13$, the cube $[3]^{13}$ has

$$3^{13} = 1{,}594{,}323 \text{ points}, \qquad 4^{13} - 3^{13} = 65{,}514{,}541 \text{ combinatorial lines},$$

and admits $3^{3^{13}} = 3^{1{,}594{,}323}$ possible colorings.

For $n = 14$, the cube $[3]^{14}$ has

$$3^{14} = 4{,}782{,}969 \text{ points}, \qquad 4^{14} - 3^{14} = 263{,}652{,}487 \text{ combinatorial lines},$$

and admits $3^{3^{14}} = 3^{4{,}782{,}969}$ possible colorings.

4.4.3 Line Clauses

Let $w \in ([3] \cup \{\star\})^n$ be a variable word, and let

$$L(w) = \{w(1), w(2), w(3)\} \subseteq [3]^n$$

be its associated combinatorial line.

Using the indexing map $\mathrm{idx}_{(3,n)} : [3]^n \to \{1, \ldots, 3^n\}$, write

$$i_a := \mathrm{idx}_{(3,n)}\Big(w(a)\Big), \qquad a \in \{1, 2, 3\}.$$

Then the points on $L(w)$ correspond to the triples of literals

$$(b_{i_1,c_1}, b_{i_1,c_2}, b_{i_1,c_3}), \quad (b_{i_2,c_1}, b_{i_2,c_2}, b_{i_2,c_3}), \quad (b_{i_3,c_1}, b_{i_3,c_2}, b_{i_3,c_3}),$$

respectively.

To forbid $L(w)$ from being monochromatic, we add the three clauses to our SAT CNF formula:

$$(\neg b_{i_1,c_1} \vee \neg b_{i_2,c_1} \vee \neg b_{i_3,c_1}), \qquad (\neg b_{i_1,c_2} \vee \neg b_{i_2,c_2} \vee \neg b_{i_3,c_2}), \qquad (\neg b_{i_1,c_3} \vee \neg b_{i_2,c_3} \vee \neg b_{i_3,c_3}).$$

The first clause rules out the all-c_1 coloring on $L(w)$ (all $b_{i_a,c_1} = 1$), the second rules out the all-c_2 assignment (all $b_{i_a,c_2} = 1$), and the third rules out the all-c_3 assignment (all $b_{i_a,c_3} = 1$). Together, they enforce that no combinatorial line in $[3]^n$ is monochromatic.

4.4.4 Additional Color Constraints

For each line $L(w)$, we must ensure that every point on $L(w)$ receives at least one color.

Accordingly, for each $a \in \{1, 2, 3\}$ we add the clause

$$(b_{i_a,c_1} \vee b_{i_a,c_2} \vee b_{i_a,c_3})$$

to the CNF formula. Since every point lies on at least one combinatorial line, adding these clauses for all lines guarantees every point receives a color.

4.4.5 Final CNF Instance

Let $\mathcal{L}$ be the set of combinatorial lines in $[3]^n$. The full formula is then

$$\begin{aligned}
\Phi_{[3]^n} = \bigwedge_{L \in \mathcal{L}} \Big[&(\neg b_{i_1,c_1} \vee \neg b_{i_2,c_1} \vee \neg b_{i_3,c_1}) \\
\wedge\ &(\neg b_{i_1,c_2} \vee \neg b_{i_2,c_2} \vee \neg b_{i_3,c_2}) \\
\wedge\ &(\neg b_{i_1,c_3} \vee \neg b_{i_2,c_3} \vee \neg b_{i_3,c_3}) \\
\wedge\ &(b_{i_1,c_1} \vee b_{i_1,c_2} \vee b_{i_1,c_3}) \\
\wedge\ &(b_{i_2,c_1} \vee b_{i_2,c_2} \vee b_{i_2,c_3}) \\
\wedge\ &(b_{i_3,c_1} \vee b_{i_3,c_2} \vee b_{i_3,c_3}) \Big].
\end{aligned}$$

Satisfiability of $\Phi_{[3]^n}$ is equivalent to the existence of a 3-coloring of $[3]^n$ that admits no monochromatic combinatorial line.

4.5 Initial Results

We first validate our approach on known values of $HJ(k; r)$. With a minor modification to the 2-color $[4]^n$ encoding to capture lines in $[3]^n$, our SAT formulation verifies the nontrivial value $HJ(3; 2) = 4$ in under one second using a single processor. It produces a monochromatic line-free coloring of $[3]^3$ and subsequently confirms that no such coloring exists for $[3]^4$.

This method is significantly faster than the original computation by Hindman and Tressler, which depended on extensive case analysis along with FPGA-assisted exhaustive search.

Applying our method naively to cases $HJ(4;2)$ and $HJ(3;3)$ yields nontrivial lower bounds

$$HJ(4;2) \geq 11 \quad \text{and} \quad HJ(3;3) \geq 9$$

after several hours of computation. After this threshold, run times increase sharply, aligning with the exponential growth of the cube size and the number of line constraints. This rapid slowdown motivates the following sections, which focus on developing approaches to improve SAT performance for finding Hales–Jewett colorings.

5 Computational Verification of Lower Bounds

In this section, we demonstrate how known van der Waerden numbers enable the machine-assisted verification of lower bounds on $HJ(4;2)$ and $HJ(3;3)$. We use van der Waerden numbers $w(4;2)$ and $w(3;3)$ to generate explicit colorings that avoid monochromatic arithmetic progressions on $[34]$ and $[26]$, respectively. We lift these colorings into the Hales–Jewett cubes $[4]^{11}$ and $[3]^{12}$, as in Section 3, and use SAT solvers to verify that these colorings admit no monochromatic combinatorial lines on their respective cubes.

While SAT is NP-complete, verifying a given assignment is a straightforward polynomial-time computation. As a result, we recover known bounds $HJ(4;2) \geq 12$ and $HJ(3;3) \geq 13$ and obtain explicit colorings for cubes $[4]^{11}$ and $[3]^{12}$ in minutes. Moreover, through an extension of the van der Waerden approach to higher dimensions, we identify a family of colorings that admit no monochromatic combinatorial line on $[3]^{13}$, thereby establishing the improved lower bound $HJ(3;3) \geq 14$.

5.1 Van der Waerden Colorings

To 2-color the cube $[4]^{11}$, we use a van der Waerden 2-coloring of $[34]$ that avoids monochromatic 4-term arithmetic progressions [5]. Concretely, we take

$$C_{VW,[34]} : [34] \to \{c_1, c_2\}, \qquad C_{VW,[34]}(i) = c_{[34]}(i),$$

where

$$\begin{aligned}(c_{[34]}(1), \ldots, c_{[34]}(34)) = (&c_1, c_2, c_1, c_2, c_2, c_2, c_1, c_1, c_1, c_2, c_1, c_2, c_2, c_1, c_2, c_2, c_2,\\ &c_1, c_1, c_1, c_2, c_1, c_1, c_2, c_1, c_2, c_2, c_2, c_1, c_1, c_1, c_2, c_1, c_2).\end{aligned}$$

Following Chvátal [5], the entries in positions

$$[34]_{\text{free}} := \{1, 12, 23, 34\}$$

are not uniquely determined. Each of these four positions may be colored c_1 or c_2, subject only to the constraint that not all four receive the same color.

We fix one such van der Waerden coloring $c_{[34]}$; the explicit choice used in our computations is the sequence displayed above.

To 3-color the cube $[3]^{12}$, we use a van der Waerden 3-coloring of [26] that avoids monochromatic 3-term arithmetic progressions [5]. We write

$$C_{VW,[26]} : [26] \to \{c_1, c_2, c_3\}, \qquad C_{VW,[26]}(i) = c_{[26]}(i),$$

where

$$(c_{[26]}(1), \ldots, c_{[26]}(26)) = (c_1, c_1, c_2, c_2, c_1, c_1, c_2, c_3, c_2, c_3, c_3, c_1, c_3, c_1, c_1, c_2, c_1, c_2, \\ c_2, c_3, c_1, c_3, c_3, c_2, c_3, c_2).$$

Here, $[26]_{\text{free}} = \emptyset$.

5.2 Lifting into the Hales–Jewett Cube

For $\mathbf{x} \in [4]^{11}$, we define the corresponding Hales–Jewett coloring by

$$C_{HJ(4;2),11}(\mathbf{x}) := C_{VW,[34]}\Big(f_{4,11}(\mathbf{x})\Big).$$

Similarly, for $\mathbf{x} \in [3]^{12}$, the Hales–Jewett coloring is

$$C_{HJ(3;3),12}(\mathbf{x}) := C_{VW,[26]}\Big(f_{3,12}(\mathbf{x})\Big).$$

For the $HJ(4;2)$ case, each function output fixes one Boolean variable b_i in the SAT CNF formula; in the $HJ(3;3)$ case, it fixes three variables $\{b_{i,c_1}, b_{i,c_2}, b_{i,c_3}\}$.

We then use SAT solvers to determine whether colorings $C_{HJ(4;2),11}(\mathbf{x})$ and $C_{HJ(3;3),12}(\mathbf{x})$ satisfy the CNF formulas described in Sections 4.3 and 4.4, respectively.

5.3 Verification of Lower Bounds

***HJ*(4; 2).** When the coloring $C_{HJ(4;2),11}(\mathbf{x})$ of $[4]^{11}$ is provided to the SAT solver, it verifies that all clauses are satisfied. Therefore, the induced coloring is a valid 2-coloring of $[4]^{11}$ that admits no monochromatic combinatorial line, verifying that

$$HJ(4;2) \geq 12.$$

***HJ*(3; 3).** Similarly, when the coloring $C_{HJ(3;3),12}(\mathbf{x})$ of $[3]^{12}$ is provided to the SAT solver, it verifies that all clauses are satisfied. Therefore, the induced coloring is a valid 3-coloring of $[3]^{12}$ that admits no monochromatic combinatorial line, verifying the bound

$$HJ(3;3) \geq 13.$$

This provides a machine-assisted verification of these classical bounds, illustrating the effectiveness of the lifting technique when paired with modern computational resources.

5.4 Extensions Beyond the Guaranteed van der Waerden Range

We aim to determine whether van der Waerden-based colorings of $[k]^n$ can be extended to $[k]^{n+1}$. To do this, we embed $[k]^n$ as an n-dimensional slice of $[k]^{n+1}$ by fixing one coordinate, color that slice according to the known Hales–Jewett coloring of $[k]^n$, and leave the remaining points uncolored. We then use a SAT solver to determine whether this partial assignment can be extended to satisfy all clauses for the encoding of $[k]^{n+1}$.

$HJ(4;2)$. Consider $\mathbf{x} \in [4]^{12}$. If $x_{12} = 1$, we assign

$$C_{HJ(4;2),12}(\mathbf{x}) = C_{HJ(4;2),11}(x_1, \dots, x_{11}).$$

In other words, we color the 11-dimensional slice

$$\{(x_1, \dots, x_{11}, 1) : (x_1, \dots, x_{11}) \in [4]^{11}\} \subseteq [4]^{12}$$

using the coloring of $[4]^{11}$. Otherwise, we leave $\mathbf{x}$ uncolored. We submit this partial coloring to a SAT solver using the CNF formula for $[4]^{12}$. After about one day of compute time, the instance was proved unsatisfiable by the solver.

$HJ(3;3)$. For the discrete cube $[3]^{13}$, the situation is notably different. If $\mathbf{x} \in [3]^{13}$ satisfies $x_{13} = 1$, we set

$$C_{HJ(3;3),13}(\mathbf{x}) = C_{HJ(3;3),12}(x_1, \dots, x_{12}).$$

This corresponds to coloring the slice

$$\{(x_1, \dots, x_{12}, 1) : (x_1, \dots, x_{12}) \in [3]^{12}\} \subseteq [3]^{13}$$

according to the coloring of $[3]^{12}$. Otherwise, we leave $\mathbf{x}$ uncolored. We submit this partial coloring to a SAT solver using the CNF formula for $[3]^{13}$. We use 15 processors in parallel to run the computation.

After approximately 24 days, totaling 360 CPU-days of compute time, the solver reports that the instance is satisfiable and provides an explicit family of 3-colorings of $[3]^{13}$ that admit no monochromatic combinatorial line. As a result, we present the improved lower bound

$$HJ(3;3) \geq 14.$$

To our knowledge, this bound on $HJ(3;3)$ is novel. Michael Nielsen also claimed that $HJ(3;3) \geq 14$ [18]; however, his argument depends on the assertion that $w(3;3) > 27$. Specifically, he references a 3-coloring of [27] with no 3-term monochromatic arithmetic progression. Since $w(3;3) = 27$, such a coloring does not exist. Therefore, the argument as stated cannot establish the bound.

Our satisfying coloring admits two free points—that is, points that can be any of the colors c_1, c_2, or c_3. Consequently, we establish nine valid Hales–Jewett colorings of $[3]^{13}$.

We have submitted the resulting 3-colorings of $[3]^{13}$ as a partial assignment to the SAT instance for $[3]^{14}$. The two free points are left unassigned. Our SAT search is highly parallel, using 220 processors and attaining an effective throughput of about seven CPU-months of compute time per day.

6 Heuristics and Search-Space Reductions

Alongside our van der Waerden-based verification and improvement of Hales–Jewett bounds, we develop heuristic methods aimed at improving SAT performance on Hales–Jewett instances.

This section documents these algorithmic methods and their outcomes. Each subsection follows a consistent format: motivation, encoding and implementation details, computational results, and discussion.

6.1 Parallelization through Line Decomposition

Motivation. The complexity of Hales–Jewett SAT instances stems from the sheer number of combinatorial lines in $[k]^n$. Our first approach is to divide the constraint set into smaller, independent subinstances that can be solved concurrently, and to build global solutions from compatible partial solutions.

Let $\mathcal{L}$ be the set of all combinatorial lines in $[k]^n$. We choose a positive integer N and partition $\mathcal{L}$ into N disjoint subsets

$$\mathcal{L}_1,\ \mathcal{L}_2,\ \ldots,\ \mathcal{L}_N, \qquad \mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_N, \qquad \mathcal{L}_i \cap \mathcal{L}_j = \emptyset \text{ for } i \neq j,$$

where each subset contains approximately the same number of lines. For each $\mathcal{L}_i$ we construct a smaller SAT instance that includes *only* the clauses for lines in $\mathcal{L}_i$. We solve these N subinstances in parallel using Python multiprocessing [19] and generate sets of partial colorings that avoid monochromatic lines within their respective line sets. Next, we compare these partial colorings across different line sets and keep only those that agree on all shared variables. Compatible partial colorings can then be merged to form a single coloring that prevents monochromaticity across the union of their line sets.

Encoding and Implementation. To partition $\mathcal{L}$ into $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_N$, we take two approaches. We algorithmically generate all combinatorial lines and use a straightforward slicing method to divide the set evenly. Alternatively, we perform a random shuffle on combinatorial lines in $\mathcal{L}$ before applying the slicing method.

For each subset $\mathcal{L}_i$, let Φ_i represent the CNF formula containing the clauses for $\mathcal{L}_i$. We solve each Φ_i using a SAT solver configured to enumerate *all* satisfying assignments. Each Φ_i is distributed to a different CPU core so that the N partial solution sets are generated in parallel. Let S_i be the set of solutions obtained for a subinstance Φ_i; then, each $s_i \in S_i$ is a Boolean assignment that satisfies Φ_i.

Given assignments $s_i \in S_i$ and $s_j \in S_j$, we call s_i and s_j compatible if they assign the same truth value to every variable in the intersection of their domains. If s_i and s_j are compatible, we form their combined assignment by taking $s_i \cup s_j$. This combined assignment satisfies $\Phi_i \cup \Phi_j$.

Repeating this compatibility-and-merge process across S_i for $1 \leq i \leq N$ produces assignments that satisfy $\Phi = \bigcup_{i=1}^{N} \Phi_i$. As a result, we obtain colorings that admit no monochromatic combinatorial line on $\mathcal{L}$.

Computational Results. Our approach reproduces all satisfying 2-colorings of $[3]^2$ in under five seconds. However, as Hales–Jewett parameters increase, exhaustively enumerating the partial sets S_i becomes prohibitively expensive.

Interestingly, our choice of method for partitioning combinatorial lines plays a significant role in SAT solver performance. Fixing $\mathcal{L}_i$s guarantees consistent SAT solve times, while partitions formed after a random shuffle vary widely: some solve in about half the time of fixed partitions, while others take roughly twice as long.

This suggests that solver performance depends not only on the number of lines, but also on which lines are selected, as different choices can simplify or complicate the problem. The mechanisms driving these speedups or slowdowns are still unclear, making this a direction for future research.

Discussion. The parallel decomposition method is effective as a solution-enumeration tool, but it does not scale to larger dimensions needed to improve Hales–Jewett bounds. The main challenge is enumeration: while SAT solvers avoid brute force by learning conflicts, this method requires the solver to list every satisfying assignment in each subproblem, thereby forcing it into a brute-force search. Additionally, the intermediate sets S_i grow large and quickly surpass memory limits, even on shared systems. Still, this method remains much faster than sequentially enumerating solutions for a single SAT instance and may have future applications.

6.2 Restrictive and Antipalindromic Colorings

Motivation. The SAT instances for cases $HJ(4;2)$ and $HJ(3;3)$ inherit significant symmetry from the corresponding discrete cubes. Many satisfying Hales–Jewett colorings are equivalent under simple cube symmetries, leading the solver to spend redundant effort exploring symmetry-equivalent solutions. We impose symmetry-breaking restrictions to reduce the search space while still allowing a large set of candidate colorings.

For the 2-colored cube $[4]^n$, we consider *antipalindromic* colorings [2]. Restricting our attention to these colorings nearly halves the solution space, but still allows most monochromatic line-free colorings to appear within it.

Definition 6.1 (Antipalindromic coloring)**.** For $\mathbf{x} = (x_1, \ldots, x_n) \in [k]^n$, let

$$\mathbf{x}^{-1} = ((k+1) - x_1, \ldots, (k+1) - x_n)$$

be its coordinatewise complement. A 2-coloring $C : [k]^n \to \{c_1, c_2\}$ is *antipalindromic* if

$$C(\mathbf{x}) \neq C(\mathbf{x}^{-1})$$

for all $\mathbf{x} \in [k]^n$, provided $\mathbf{x} \neq \mathbf{x}^{-1}$.

The cube is partitioned into complement-pairs $\{\mathbf{x}, \mathbf{x}^{-1}\}$. If k is odd, there is a unique self-inverse point $\mathbf{x_0} = \left(\frac{k+1}{2}, \ldots, \frac{k+1}{2}\right)$; we place no restrictions on $\mathbf{x_0}$. For even k, there are no self-inverse points.

We leave it to the reader to verify that the van der Waerden coloring $C_{VW,[34]}$ from Section 5.1 is antipalindromic with respect to the reflection $i \mapsto i^{-1} := 35 - i$; that is,

$$C_{VW,[34]}(i) \neq C_{VW,[34]}(35 - i) \quad \text{for all } i \in [34].$$

Antipalindromic colorings are a specific type of *half-coloring*, introduced in [3]. A half-coloring is a coloring where the two color classes differ in size by at most one. Every antipalindromic coloring naturally satisfies this condition: each complement-pair $\{\mathbf{x}, \mathbf{x}^{-1}\}$ contributes one point to each color class. If k is odd, there is a unique self-inverse point $\mathbf{x_0} = \left(\frac{k+1}{2}, \ldots, \frac{k+1}{2}\right)$, which is not paired with a distinct complement; this singleton point may be assigned either color. Hence the two color classes differ in size by at most one, so antipalindromic colorings form a subclass of half-colorings.

Antipalindromic colorings capture many, but not all valid colorings of the discrete cube. Therefore, this strategy does not aim to compute an exact Hales–Jewett number, but rather to find a coloring that can improve a lower bound.

Encoding and Implementation. In the SAT encoding of $HJ(4;2)$, $\mathbf{x} \in [4]^n$ is assigned the variable $b_i \in \{0,1\}$ where

$$i = \mathrm{idx}_{(4,n)}(\mathbf{x}).$$

For any point $\mathbf{x}$, let $\mathbf{x}^{-1}$ denote its inverse, and define the index of the inverse point by

$$i^{-1} \; = \; \mathrm{idx}_{(4,n)}(\mathbf{x}^{-1}).$$

Because we consider $k = 4$, the complement map $\mathbf{x} \mapsto \mathbf{x}^{-1}$ has no fixed points. Every $\mathbf{x}$ lies in a pair $\{\mathbf{x}, \mathbf{x}^{-1}\}$, and we can identify one variable per pair without special handling.

To exploit the symmetry between $\mathbf{x}$ and $\mathbf{x}^{-1}$, we impose the constraint

$$b_i \neq b_{i^{-1}}.$$

Rather than encoding this constraint explicitly, we keep only one variable per inverse pair: specifically, we retain the variable with the smaller index, $b_{\min(i,i^{-1})}$. The other variable is eliminated and replaced everywhere by its negation.

Thus, if $i < i^{-1}$, we substitute $b_{i^{-1}}$ by $\neg b_i$ (and symmetrically if $i^{-1} < i$). This makes $b_i \neq b_{i^{-1}}$ hold automatically, while halving the number of variables in the SAT CNF formula.

Computational Results. The search for antipalindromic colorings of $[4]^{12}$ is still ongoing. As of December 2025, a SAT solver, distributed across 15 processors, has been running the computation for more than one CPU-year of compute time and has yet to terminate. Hence, the outcome of this method remains unknown.

Discussion. Enforcing antipalindromicity presents a clear structural reduction by removing approximately half of the free variables from the search space. We note that analyzing antipalindromic colorings of $[4]^{12}$ remains computationally challenging, as demonstrated by the results above. This approach is best regarded as a modular restriction that can be combined with additional reductions or solver-guidance techniques in future research.

6.3 Geometric Lines

Motivation. The following approach aims to establish Hales–Jewett lower bounds by considering a broader class of lines, referred to as geometric lines.

Our definition of a geometric line is consistent with that defined in Section 2.

Let $HJ^*(k;r)$ denote the *geometric Hales–Jewett number*, defined as the least n such that every r-coloring of $[k]^n$ admits a monochromatic geometric line. Because forbidding geometric lines is a more restrictive requirement than forbidding combinatorial lines, we have

$$HJ^*(k;r) \leq HJ(k;r).$$

Therefore, a SAT instance that proves $HJ^*(k;r) \geq n$ also shows that $HJ(k;r) \geq n$. Geometric lines create a denser set of constraints on the cube $[k]^n$, which may enforce monochromatic structures more quickly and result in a more direct search space. This motivates our decision to use them for lower-bound searches.

Encoding and Implementation. To encode monochromatic geometric line avoidance constraints for $[k]^n$, we generate all geometric variable words of length n and their corresponding geometric lines $L^*(w^*)$. Our CNF clause generation is consistent with Section 4. There are

$$(k+2)^n - k^n$$

geometric variable words on $[k]^n$. Each coordinate can use any of the k fixed symbols or the two variables $\star, \diamond$. Purely fixed words (without $\star$ or $\diamond$), of which there are k^n, are excluded.

For the dimensions of interest, this yields:

$$6^{12} - 4^{12} = 2{,}160{,}005{,}120 \quad \text{geometric variable words for } [4]^{12},$$

$$5^{14} - 3^{14} = 6{,}098{,}732{,}656 \quad \text{geometric variable words for } [3]^{14}.$$

The geometric Boolean formulas are considerably larger than their combinatorial counterparts.

Computational Results. The SAT solver search for monochromatic geometric line-free colorings of $[4]^{12}$ is still in progress. The instance is running on 30 processors and, as of December 2025, has used 22 CPU-days of compute time without termination. Currently, the CNF formula for $[3]^{14}$ is too large to parse.

Discussion. Geometric lines provide a framework that may yield lower bounds for Hales–Jewett numbers. While the complete geometric encoding requires more computational resources than the standard combinatorial approach, it is possible that this computational process yields a lower bound on $HJ^*(4;2)$, and hence on $HJ(4;2)$. Additional computational resources are required to address the geometric line encoding for $[3]^{14}$.

6.4 Minimal Hales–Jewett Sets and Upper-Bound Heuristics

Motivation. The following method aims to establish upper bounds on $HJ(4;2)$. To prove that $HJ(4;2) \leq n$, one must show that every 2-coloring of $[4]^n$ admits a monochromatic combinatorial line. Using naive SAT methods to determine this result is computationally infeasible, even for moderate values of n. A more effective approach is to identify a smaller set of unavoidable lines. If a subset of line constraints results in an unsatisfiable SAT instance, then no 2-coloring can avoid monochromaticity on the entire line set, proving $HJ(4;2) \leq n$.

This idea is closely related to *minimal Hales–Jewett sets*, introduced by Hindman and Jordan [11]. A subset $A \subseteq [k]^n$ is a Hales–Jewett set if every coloring of A admits a monochromatic combinatorial line, and it is minimal if no proper subset has this property. While we do not attempt to classify minimal sets here, their existence encourages the search among unavoidable subsets of lines to establish upper bounds.

We restrict our attention to points that lie on the most combinatorial lines, because SAT clauses generated by these points create more structure than those generated by points on fewer lines. We seek to establish whether the set of lines through a high-incidence point is already monochromatically unavoidable in some dimension. To determine which points have maximal incidence, we count the number of combinatorial lines passing through an arbitrary point.

Theorem 6.2. *Let $k, n \in \mathbb{Z}_{>0}$ and consider the discrete cube $[k]^n$. Fix a point $\mathbf{x} = (x_1, \ldots, x_n) \in [k]^n$. For each $a \in [k]$, define*

$$I_a = \{\, i \in [n] : x_i = a \,\}, \qquad m_a = |I_a|.$$

Then the number of combinatorial lines in $[k]^n$ that pass through $\mathbf{x}$ is

$$\sum_{a=1}^{k} (2^{m_a} - 1).$$

Proof. A combinatorial line $L(S, \mathbf{x_0})$ passes through $\mathbf{x}$ if and only if there exists $a \in [k]$ such that $x_i = a$ for all $i \in S$ and $x_i = x_{0_i}$ otherwise. Thus, S must be a nonempty subset of

$$I_a = \{\, i : x_i = a \,\}.$$

For each $a \in [k]$, there are $2^{m_a} - 1$ nonempty subsets $S \subseteq I_a$, each corresponding to a distinct line through $\mathbf{x}$. Summing over all $a \in [k]$ gives

$$\#\{\text{lines through } \mathbf{x}\} = \sum_{a=1}^{k} (2^{m_a} - 1).$$

□

Corollary 6.3. *If all coordinates of* $\mathbf{x}$ *are distinct, then* $m_a = 0$ *or* $m_a = 1$ *for all* $a \in [k]$ *and the number of lines through* $\mathbf{x}$ *is at most* n*. If all coordinates of* $\mathbf{x}$ *are identical, then* $m_a = n$ *for some* $a \in [k]$ *and all other* $m_b = 0$*, giving* $2^n - 1$ *lines, which is maximal.*

By Corollary 6.3, the points with identical coordinates are incident to the greatest number of combinatorial lines. Accordingly, we fix the point

$$\mathbf{1} = (1, 1, \ldots, 1),$$

which we call the *origin*.

Let $\mathcal{L}_{\mathbf{1}}$ denote the set of combinatorial lines passing through $\mathbf{1}$. In dimension n, this family has size

$$|\mathcal{L}_{\mathbf{1}}| = 2^n - 1.$$

We then consider the CNF formula obtained by including clauses only for lines in $\mathcal{L}_{\mathbf{1}}$.

Encoding and Implementation. Each line $L_{\mathbf{1}} \in \mathcal{L}_{\mathbf{1}}$ is generated by a variable word in $\{1, \star\}^n$. We generate our SAT CNF formula by appending clauses for each line through the origin:

$$\Phi_{\mathbf{1}} := \bigwedge_{L_{\mathbf{1}} \in \mathcal{L}_{\mathbf{1}}} \Big[(\neg b_{i_1} \vee \cdots \vee \neg b_{i_k}) \wedge (b_{i_1} \vee \cdots \vee b_{i_k}) \Big],$$

where $(i_1, \ldots, i_k)$ are the Boolean indices assigned to the points of $L_{\mathbf{1}}$.

In this case, the usual base-k indexing $\mathrm{idx}_{(k,n)}$ disperses the points in $\mathcal{L}_{\mathbf{1}}$ across the Boolean indices $\{1, 2, \ldots, k^n\}$, resulting in poor locality in the CNF formula. A solver that receives Boolean indices $\{b_1, b_2, b_3\}$ performs significantly better than one that receives $\{b_1, b_6, b_7\}$, because it considers $\{b_2, b_3, b_4, b_5\}$ as free variables that it must also evaluate.

To preserve the contiguous nature of the Boolean indices, we introduce a new function

$$\mathrm{idx}_{(k,n),\mathbf{1}} : \bigcup_{L_{\mathbf{1}} \in \mathcal{L}_{\mathbf{1}}} L_{\mathbf{1}} \to \mathbb{Z}_{>0}.$$

that orders points on the cube line-by-line outward from $\mathbf{1}$.

Let $L_{\mathbf{1}}(w) \in \mathcal{L}_{\mathbf{1}}$. Then $L_{\mathbf{1}}(w) = \{\mathbf{1}, w(2), \ldots, w(k)\}$, where $L_{\mathbf{1}}(w)$ is uniquely determined by the variable word

$$w = (w_1, \ldots, w_n) \in \{1, \star\}^n.$$

We encode w as a binary string by setting

$$w'_i = \begin{cases} 1 & \text{if } w_i = \star, \\ 0 & \text{if } w_i = 1, \end{cases}$$

and interpret $(w'_1, \ldots, w'_n)$ as the corresponding base-2 integer via

$$\mathrm{bin}(w) = \sum_{i=1}^{n} w'_i \, 2^{n-i}.$$

Hence, $\mathrm{bin}(w) \in \{1, \ldots, 2^n - 1\}$ assigns a unique value to each combinatorial line.

If $\mathbf{x} \in L(w)$, then its position along the line is recorded by $\max(\mathbf{x}) \in \{1, \ldots, k\}$. We then define

$$\mathrm{idx}_{(k,n),\mathbf{1}}(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} = \mathbf{1}, \\ (\mathrm{bin}(w) - 1)(k-1) + \max(\mathbf{x}), & \mathbf{x} \text{ lies on origin line } L(w). \end{cases}$$

The function $\mathrm{idx}_{(k,n),\mathbf{1}}$ assigns each origin line a contiguous block of $(k-1)$ indices for all non-origin points, and each point one of these $(k-1)$ indices. This method enhances solver locality and is essential for practical use.

Computational Results. We used SAT solvers to determine the satisfiability of origin-line CNF formulas encoding $[4]^n$ for all $n \leq 26$. The solver identified a valid coloring in each case. Higher-dimensional tests were not feasible due to the size of the resulting CNF formulas.

Discussion. Theorem 6.2 provides a clear structural reason to consider identical-coordinate points: they maximize line incidence and, as a result, boost the immediate constraint pressure of a reduced SAT encoding. While this approach has not yet established an upper bound for $HJ(4;2)$, it has the potential to do so, and we outline a path forward in Section 8.

7 Off-diagonal Hales–Jewett Numbers

In this section, we introduce a new off-diagonal version of the Hales–Jewett problem. Unlike the classical cases, which require preventing full monochromatic lines in every color, the off-diagonal parameters enforce asymmetric avoidance rules: one color must avoid monochromatic lines, while others must avoid shorter monochromatic segments.

For a fixed color, forbidding short monochromatic segments is a stronger constraint than forbidding only full monochromatic lines. These tighter constraints often make the SAT encodings easier for solvers, because conflicts are detected earlier in the search. This makes the off-diagonal parameters especially well suited to SAT-based computation.

7.1 Definitions and Bounds

We first introduce *partial combinatorial lines.*

Definition 7.1 (Partial combinatorial line)**.** Let $n, k \in \mathbb{Z}_{>0}$, and let $L(w) = \{w(a) : a \in [k]\}$ be a (full) combinatorial line in $[k]^n$. A *partial combinatorial line of length m* is any subset of the form

$$\{\, w(a) : a \in \{i+1, i+2, \ldots, i+m\} \,\},$$

for $0 \leq i \leq k - m$.

We now formally introduce the two- and three-colored off-diagonal Hales–Jewett numbers.

Definition 7.2 (Off-diagonal Hales–Jewett numbers)**.** Let $k, m \in \mathbb{Z}_{>0}$ with $m \leq k$. The *2-color off-diagonal Hales–Jewett number* $HJ(k, m; 2)$ is defined as the smallest positive integer n such that every 2-coloring of $[k]^n$ admits at least one of the following:

1. a c_1-monochromatic combinatorial line of length k, or
2. a c_2-monochromatic partial combinatorial line of length m.

For $j \leq m \leq k$, the *3-color off-diagonal Hales–Jewett number* $HJ(k, m, j; 3)$ is the smallest positive integer n such that every 3-coloring of $[k]^n$ contains at least one of:

1. a c_1-monochromatic combinatorial line of length k,
2. a c_2-monochromatic partial combinatorial line of length m, or
3. a c_3-monochromatic partial combinatorial line of length j.

Bounds. By definition, the off-diagonal Hales–Jewett numbers are bounded above by their classical counterparts. In particular, for any $m \leq k$,

$$HJ(k, m; 2) \leq HJ(k; 2).$$

Indeed, if $HJ(k; 2) = n$, then every 2-coloring of $[k]^n$ admits a monochromatic combinatorial line of length k. If this line has color c_1, the off-diagonal condition is satisfied immediately. If it has color c_2, then it contains a length-m line segment of color c_2 (since $m \leq k$), so the off-diagonal condition is again satisfied.

An analogous argument shows that

$$HJ(3, 2, 2; 3) \ \leq \ HJ(3, 3, 2; 3) \ \leq \ HJ(3; 3).$$

7.2 SAT Encoding for $HJ(4, m; 2)$

The SAT encodings for $HJ(4, 2; 2)$ and $HJ(4, 3; 2)$ are straightforward modifications of the standard $HJ(4; 2)$ construction. All conventions follow Section 4, except for the SAT CNF clause generation.

Let $L \subseteq [4]^n$ be a combinatorial line, and let $(b_{i_1}, b_{i_2}, b_{i_3}, b_{i_4})$ denote the Boolean variables obtained by applying $\mathrm{idx}_{(4,n)}$ to its four points.

To forbid a c_1-monochromatic line of length 4, we include the clause

$$(\neg b_{i_1} \vee \neg b_{i_2} \vee \neg b_{i_3} \vee \neg b_{i_4}),$$

which rules out the all-c_1 assignment on L.

7.2.1 $HJ(4, 2; 2)$

To forbid a c_2-monochromatic *partial* combinatorial line of length 2, we add the clauses

$$(b_{i_1} \vee b_{i_2}), \qquad (b_{i_2} \vee b_{i_3}), \qquad (b_{i_3} \vee b_{i_4}),$$

each of which prevents a pair of consecutive points on L from both being colored c_2.

7.2.2 $HJ(4, 3; 2)$

To forbid a c_2-monochromatic partial combinatorial line of length 3, we include clauses

$$(b_{i_1} \vee b_{i_2} \vee b_{i_3}), \qquad (b_{i_2} \vee b_{i_3} \vee b_{i_4}),$$

which prevent any triple of consecutive points on L from all being colored c_2.

Adding the relevant clauses to the CNF formula yields the encodings for $HJ(4, 2; 2)$ and $HJ(4, 3; 2)$.

7.3 SAT Encoding for $HJ(3, m, j; 3)$

Recall from Section 4.4 that each point $\mathbf{x} \in [3]^n$ is assigned three Boolean variables

$$(b_{i,c_1}, b_{i,c_2}, b_{i,c_3}),$$

where $b_{i,c_k} = 1$ if $\mathbf{x}$ receives color c_k.

Let $L \subseteq [3]^n$ be a combinatorial line, and let

$$\{(b_{i_1,c_1}, b_{i_1,c_2}, b_{i_1,c_3}), (b_{i_2,c_1}, b_{i_2,c_2}, b_{i_2,c_3}), (b_{i_3,c_1}, b_{i_3,c_2}, b_{i_3,c_3})\}$$

denote the triples of variables assigned to the three points of L by $\mathrm{idx}_{(3,n)}$.

To forbid a c_1-monochromatic line of length 3, we include

$$(\neg b_{i_1,c_1} \vee \neg b_{i_2,c_1} \vee \neg b_{i_3,c_1}),$$

which prevents the all-c_1 assignment on L.

To guarantee that every point receives a color, we also include the clauses

$$(b_{i_1,c_1} \vee b_{i_1,c_2} \vee b_{i_1,c_3}), \qquad (b_{i_2,c_1} \vee b_{i_2,c_2} \vee b_{i_2,c_3}), \qquad (b_{i_3,c_1} \vee b_{i_3,c_2} \vee b_{i_3,c_3}),$$

as in Section 4.4.

7.3.1 $HJ(3, 2, 2; 3)$

To forbid a c_2-monochromatic partial line of length 2, we include

$$(\neg b_{i_1,c_2} \vee \neg b_{i_2,c_2}), \qquad (\neg b_{i_2,c_2} \vee \neg b_{i_3,c_2}),$$

preventing any consecutive pair of points on L from both being colored c_2.

Similarly, to forbid a c_3-monochromatic partial line of length 2, we include

$$(\neg b_{i_1,c_3} \vee \neg b_{i_2,c_3}), \qquad (\neg b_{i_2,c_3} \vee \neg b_{i_3,c_3}),$$

which prevents consecutive pairs of points on L from both being colored c_3.

7.3.2 $HJ(3,3,2;3)$

To forbid a c_2-monochromatic partial line of length 3, we add the clause

$$(\neg b_{i_1,c_2} \vee \neg b_{i_2,c_2} \vee \neg b_{i_3,c_2}),$$

excluding the all-c_2 assignment on L.

Then, as in the $HJ(3,2,2;3)$ case, we include

$$(\neg b_{i_1,c_3} \vee \neg b_{i_2,c_3}), \qquad (\neg b_{i_2,c_3} \vee \neg b_{i_3,c_3})$$

to prevent consecutive pairs of points on L from both being colored c_3.

These clauses yield the SAT instances for $HJ(3,2,2;3)$ and $HJ(3,3,2;3)$, respectively.

7.4 Off-Diagonal Results

Within our encoding framework, we use a SAT solver to prove that

$$HJ(3,2;2)=3, \qquad HJ(4,2;2)=4, \qquad HJ(5,2;2)=5, \qquad HJ(6,2;2)=6.$$

These values motivate the following conjecture.

Conjecture 7.3. *Let $k \in \mathbb{Z}_{>0}$ such that $k \geq 2$. Then*

$$HJ(k,2;2)=k.$$

We have proven that $HJ(4,3;2) > 7$. It remains to be determined whether $HJ(4,3;2) = 8$. As of December 2025, the computation has run for 434 CPU-days and has not yet terminated.

For the ternary case, we proved that

$$HJ(3,2,2;3)=5,$$

with the solver terminating in under one second.

In addition, we have established that $HJ(3,3,2;3) > 8$. It remains to be determined whether $HJ(3,3,2;3) = 9$. As of December 2025, the computation has run for 117 CPU-days and has not yet terminated.

8 Conclusions and Future Directions

This section summarizes the main outcomes of this paper and outlines several directions for future work suggested by our methods and results.

8.1 Conclusions

We developed a new framework for understanding and computing Hales–Jewett numbers, focusing on $HJ(4;2)$ and $HJ(3;3)$. We mapped points in the discrete cube to Boolean variables and formulated the search for monochromatic line-free colorings as a SAT problem.

Combining this encoding method with known van der Waerden numbers and modern computational techniques, we obtained SAT-based, machine-assisted proofs of known lower bounds on $HJ(4;2)$ and $HJ(3;3)$. Extending these computations to higher dimensions yielded an improved bound for the ternary case. In particular, we established that

$$HJ(3;3) \geq 14.$$

We introduced several algorithmic techniques to reduce the SAT search space size and complexity. These include parallelization through line decomposition, symmetry breaking via antipalindromic colorings, restrictions to geometric lines, and a heuristic based on high-incidence points. We also proposed off-diagonal Hales–Jewett numbers as a computationally accessible variant of the classical problem, and proved several exact values and bounds in this setting.

8.2 Future Directions

The methods and results above suggest several immediate extensions.

First, a number of our computations are still in progress. At present, five experiments remain underway: antipalindromic colorings of $[4]^{12}$, monochromatic geometric line-free colorings of $[4]^{12}$, the extension of satisfying $[3]^{13}$ colorings to $[3]^{14}$, and the determination of whether $HJ(4,3;2) = 8$ and $HJ(3,3,2;3) = 9$. As these runs complete, we expect additional bounds or exact values to follow directly from our existing framework.

Second, many of the techniques developed to address $HJ(4;2)$ transfer to the $HJ(3;3)$ setting with minimal conceptual change. In particular, geometric line-based encodings (Section 6.3) and the study of high-incidence points (Section 6.4) appear promising in the ternary regime. We also seek to determine whether an analog of antipalindromic symmetry breaking can be formulated for a 3-colored cube.

Third, our high-incidence origin-point algorithm motivates a broader refinement. Fix a dimension n, then iteratively choose high-incidence points in $[k]^n$. For each chosen point, build a CNF formula from the lines passing through it and use a SAT solver to search for satisfying colorings. By progressively enlarging the constraint set with lines that pass through additional high-incidence points, we may obtain a scalable route to stronger bounds.

Fourth, results in Section 6.1 show that SAT performance depends on the specific combinatorial lines included, not merely their quantity, motivating heuristics for subset selection and ordering.

Finally, several directions remain open for the off-diagonal Hales–Jewett numbers. Beyond resolving the two values currently under computation, a central objective is to prove Conjecture 7.3, thereby placing the observed pattern on a formal footing.

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