Computers Meet Combinatorics: SAT Solvers and the Search for Hales–Jewett Bounds

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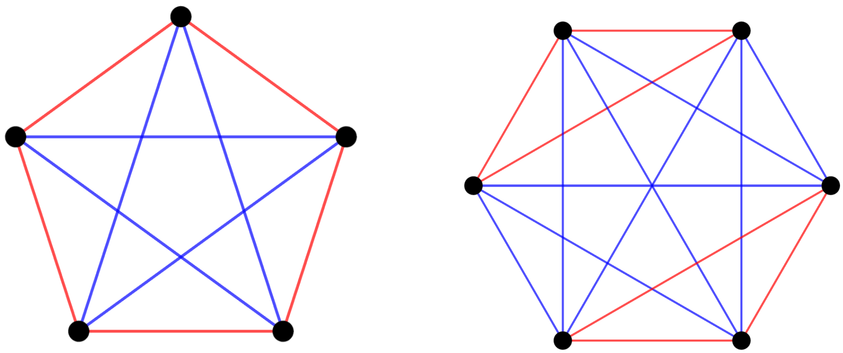
Outline

Ramsey Theory, Van der Waerden, and Hales–Jewett

What is Ramsey Theory?

Ramsey theory studies when structure becomes unavoidable in sufficiently large systems.

Example: Suppose we color each edge of K_n red or blue. How large must n be to force a monochromatic triangle in every coloring of K_n?



Van der Waerden Theorem and Numbers

We now consider **Ramsey Theory in the integers**.

Theorem (Van der Waerden Theorem [2])

For integers $k, r \in \mathbb{Z}^+$, there exists an integer $W(k; r)$ such that for all $N \geq W(k; r)$, every r-coloring of $\{1, \ldots, N\}$ contains a monochromatic k-term arithmetic progression (i.e. a sequence $\{a, a + d, \ldots, a + (k-1)d\}$).

The **Van der Waerden Theorem** guarantees such an N exists, and the **Van der Waerden number $W(k; r)$** is the smallest such N for given k and r.

EXAMPLE: $W(3,2) = 9$

Van der Waerden number: $W(3,2) = 9$

We can 2-color $\{1, \ldots, 8\}$ to avoid any monochromatic 3-term arithmetic progression:

$$\{1, 2, 3, 4, 5, 6, 7, 8\}$$

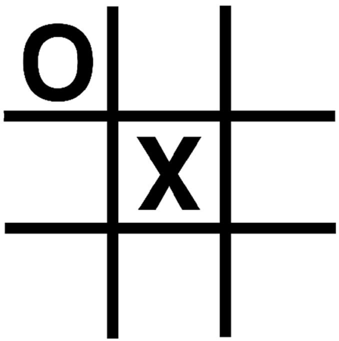
This coloring avoids all monochromatic progressions of the form:

$$\{a, a+d, a+2d\}$$

However, no 2-coloring of $\{1, \ldots, 9\}$ can avoid a monochromatic 3-term arithmetic progression.

Hales–Jewett in Two Dimensions

In two dimensions, the Hales–Jewett problem looks like a game of tic-tac-toe. Can we color the board in a way that avoids a monochromatic winning line?



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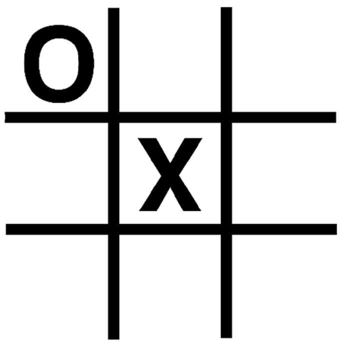
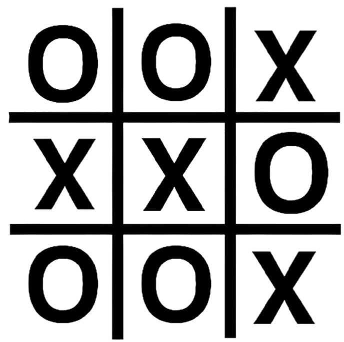
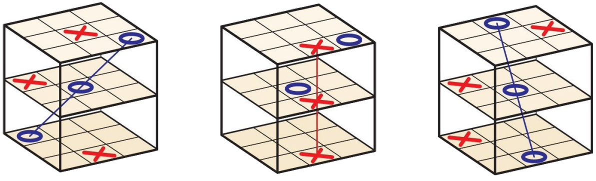




Hales–Jewett in Three Dimensions

What about the $3 \times 3 \times 3$ cube?



This problem becomes significantly harder, as there are

$$2^{27} = 134{,}217{,}728 \text{ possible colorings.}$$

From Grids to Higher Dimensions

To make our question precise, we need to formalize **points and lines on hypercubes**.

Let $[k] = \{1, 2, \ldots, k\}$ be an alphabet.

Words of length n are sequences

$$[k]^n = \{(x_1, \ldots, x_n) : x_i \in [k]\}.$$

We think of $[k]^n$ as a k-ary n-dimensional discrete cube.

Defining Lines in Higher Dimensions

To describe lines, we introduce **variable words**.

A **variable word** is a sequence where some coordinates are replaced by a wildcard $\star$:

$$w = (w_1, \ldots, w_n) \in ([k] \cup \{\star\})^n.$$

Combinatorial Lines

Fix a variable word w. Substituting a letter $a \in [k]$ into all wildcard positions gives

$$w(a) = (u_1, \ldots, u_n), \qquad u_i = \begin{cases} a & \text{if } i \in S(w), \\ w_i & \text{if } i \notin S(w). \end{cases}$$

The set

$$L(w) = \{w(a) : a \in [k]\}$$

is called a **combinatorial line**.

This is our **line** for the Hales–Jewett numbers.

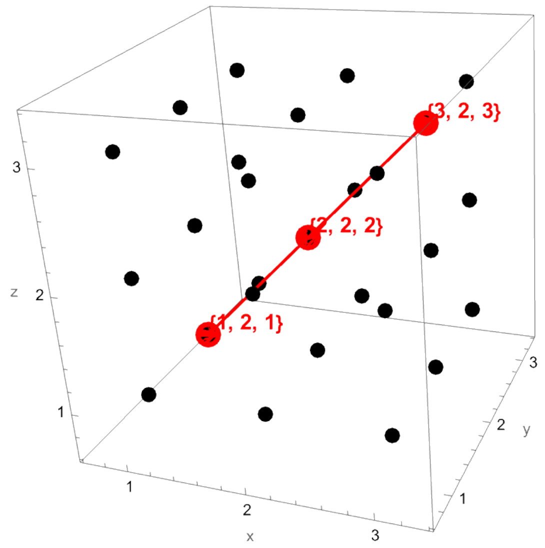
An Example

Example: Let $[3] = \{1, 2, 3\}$ and set

$$w = (\star, 2, \star).$$

Substituting $a \in [3]$ gives

$$L(w) = \{(1, 2, 1), (2, 2, 2), (3, 2, 3)\}.$$



The Hales–Jewett Theorem

Question. Once the cube dimension becomes sufficiently large, do monochromatic combinatorial lines (i.e. winning lines) become unavoidable?

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Theorem (Hales–Jewett [1])

For every $k, r \in \mathbb{Z}^+$, there exists an integer n such that for all $d \geq n$, every r-coloring of $[k]^d$ contains a monochromatic combinatorial line.

Hales–Jewett Numbers

Definition

The *Hales–Jewett number* $HJ(k; r)$ is the smallest integer n such that every r-coloring of $[k]^n$ contains a monochromatic combinatorial line.

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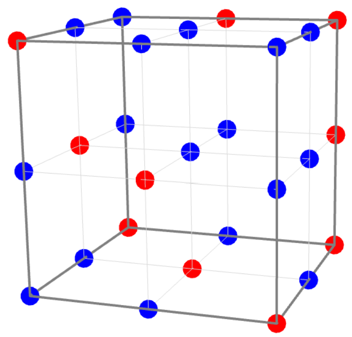
How **large** are these numbers?

Exact Values

Very little is known about exact Hales–Jewett numbers. The **only** nontrivial value known exactly is

$$HJ(3;2) = 4.$$



Best Known Bounds

For the next simplest cases, we only have bounds:

$$12 \leq HJ(4;2) \leq 10^{11}, \qquad 13 \leq HJ(3;3) \leq 2^{2^{3c}},$$

for some constant c.

Why Brute Force Fails

Case $HJ(4;2) \geq 12$**:**

- Number of colorings of $[4]^{12}$ is $2^{4^{12}} = 2^{16{,}777{,}216}$
- Number of combinatorial lines: 227,363,409

Case $HJ(3;3) \geq 13$**:**

- Number of colorings of $[3]^{13}$ is $3^{3^{13}} \approx 2^{2{,}526{,}942}$
- Number of combinatorial lines is 65,514,541

Exhaustive search is **impossible**, so we need a more efficient way to search.

Boolean Satisfiability (SAT)

Boolean Formulas

A **Boolean formula** is a logical expression built from binary variables. Equivalently, it defines a function

$$f : \{0, 1\}^n \to \{0, 1\}.$$

We combine variables using logical operations:

$$\wedge \text{ (AND)}, \qquad \vee \text{ (OR)}, \qquad \neg \text{ (NOT)}$$

Example:

$$(b_1 \vee \neg b_2) \;\wedge\; (\neg b_3 \vee b_4 \vee \neg b_5)$$

Conjunctive Normal Form (CNF)

We require formulas to be in **conjunctive normal form (CNF).**

A CNF formula is a conjunction of clauses, where each clause is a disjunction of literals.

Example:

$$(b_1 \vee \neg b_2) \ \wedge \ (\neg b_3 \vee b_4 \vee \neg b_5)$$

The SAT Problem

Input: A Boolean formula in conjunctive normal form (CNF)

Question: Does there exist an assignment of values to its variables that makes the formula true?

If such an assignment exists, the formula is **satisfiable**; otherwise, it is **unsatisfiable**.

Why SAT Solvers Work

SAT is NP-complete, yet modern solvers are **remarkably effective** in practice.

They routinely handle formulas with **millions** of variables and clauses, allowing us to explore vast combinatorial spaces efficiently.

How SAT Solvers Work

Example:

$$(\neg b_1 \vee b_3) \ \wedge \ (\neg b_2 \vee b_4) \ \wedge \ (\neg b_3 \vee \neg b_4)$$

Step 1: Assign $b_1 = 1$, $b_2 = 1$

Step 2: Propagation forces

$$b_3 = 1, \quad b_4 = 1$$

Step 3: Conflict!

$(\neg b_3 \vee \neg b_4)$ is false

The Key Idea: Learning

From the conflict, the solver learns:

$$b_1 = 1 \text{ and } b_2 = 1 \Rightarrow \text{contradiction}$$

It adds a new clause:

$$(\neg b_1 \vee \neg b_2)$$

Key insight: The solver learns from conflicts, avoiding repeated mistakes and pruning the search space.

SAT Solvers and Combinatorial Structure

The performance of SAT solvers depends strongly on the **problem structure.**

$HJ(3;3) \geq n$	Incidence-aware	Standard heap
7	77	134
8	307	386
9	1281	3288

Conflict Counts on Hales–Jewett SAT Instances

Key Insight: Combinatorial structure strongly influences solver behavior.

Encoding Hales–Jewett as SAT

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Goal. For fixed $k, r, n \in \mathbb{Z}^+$, determine whether there exists a coloring

$$[k]^n \to \mathcal{C}_r$$

with no monochromatic combinatorial line.

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$$[k]^n \to \mathcal{C}_r$$

with no monochromatic combinatorial line.

Idea. Create a Boolean formula $\Phi_{[k]^n}$ such that:

- $\Phi_{[k]^n}$ satisfiable $\Rightarrow HJ(k; r) \geq n + 1$
- $\Phi_{[k]^n}$ unsatisfiable $\Rightarrow HJ(k; r) \leq n$

Encoding $HJ(4;2)$: Points

Each point $x \in [4]^n$ is assigned a variable b_x:

$$b_x = 1 \iff x \text{ is red}, \qquad b_x = 0 \iff x \text{ is blue}.$$

Each variable represents the color of a point.

ENCODING $HJ(4;2)$: LINES

Let $L = \{x(1), x(2), x(3), x(4)\} \subseteq [4]^n$ be a combinatorial line.

To ensure L is not monochromatic, we enforce:

$$(\neg b_{x(1)} \vee \neg b_{x(2)} \vee \neg b_{x(3)} \vee \neg b_{x(4)}) \quad \wedge \quad (b_{x(1)} \vee b_{x(2)} \vee b_{x(3)} \vee b_{x(4)})$$

Applying this to all lines $L \in \mathcal{L}$ yields a formula $\Phi_{[4]^n}$, which is satisfiable if and only if $HJ(4;2) > n$.

ENCODING $HJ(3;3)$

In the 3-color case, each point $x \in [3]^n$ is assigned three variables:

$$(b_{x,r}, b_{x,b}, b_{x,g})$$

$$b_{x,r} = 1 \iff x \text{ is red}, \quad b_{x,b} = 1 \iff x \text{ is blue}, \quad b_{x,g} = 1 \iff x \text{ is green}$$

Each triple of variables represents the color of a point. Each point has one (or more) colors.

ENCODING $HJ(3;3)$

Let $L = \{x(1), x(2), x(3)\}$ be a combinatorial line.

To prevent monochromatic lines, we add:

$$\begin{aligned} & (\neg b_{x(1),r} \vee \neg b_{x(2),r} \vee \neg b_{x(3),r}) \\ \wedge \quad & (\neg b_{x(1),b} \vee \neg b_{x(2),b} \vee \neg b_{x(3),b}) \\ \wedge \quad & (\neg b_{x(1),g} \vee \neg b_{x(2),g} \vee \neg b_{x(3),g}) \end{aligned}$$

We also require each point to receive a color:

$$(b_{x,r} \vee b_{x,b} \vee b_{x,g})$$

The resulting formula is satisfiable $\iff HJ(3;3) > n$.

An Improved Hales–Jewett Bound

Main Result

Theorem (Conlon, F-, Robertson)

$$HJ(3;3) \geq 14.$$

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Theorem (Conlon, F-, Robertson)

$$\boldsymbol{HJ(3;3) \geq 14.}$$

Proof idea: We construct a coloring of $[3]^{13}$ in two stages:

1. Use a van der Waerden coloring to build a structured coloring of $[3]^{12}$
2. Extend this partial coloring to $[3]^{13}$ using a SAT solver

Building the Construction

We begin with a 3-coloring

$$C_{VW} : [1, 26] \to \{\text{red}, \text{blue}, \text{green}\}$$

with no monochromatic 3-term arithmetic progression.

This exists since

$$W(3; 3) = 27.$$

We use this to construct a monochromatic line-avoidant coloring of $[3]^{12}$.

Coloring $[3]^{12}$ from van der Waerden

Define $f : [3]^{12} \to [1, 25]$,

$$f(x) = 1 + \sum_{i=1}^{12} (x_i - 1).$$

We color each point $x \in [3]^{12}$ by

$$C_{12}(x) := C_{VW}(f(x)).$$

Interpretation: Each point in the cube is colored via its image under f.

Why Lines Map to Progressions

Let $L = \{w(1), w(2), w(3)\}$ be a combinatorial line arising from variable word w.

If w has t wildcard positions, then

$$f(w(2)) - f(w(1)) = t, \quad f_(w(3)) - f(w(2)) = t.$$

Thus, the images form a **3-term arithmetic progression** (with common difference t).

Why This Prevents Monochromatic Lines

Suppose a line $L = \{w(1), w(2), w(3)\}$ were monochromatic.

Then

$$C_{VW}(f(w(1))) = C_{VW}(f(w(2))) = C_{VW}(f(w(3))).$$

These these values form a 3-term arithmetic progression - contradiction!

Thus, **no monochromatic lines exist** in $[3]^{12}$ under C_{12}.

Extending to $[3]^{13}$

We view $[3]^{13}$ as three 12-dimensional slices and fix one slice,

$$\{(x_1, \dots, x_{12}, 1) : (x_1, \dots, x_{12}) \in [3]^{12}\}.$$

Color this slice using C_{12}:

$$C_{13}(x_1, \dots, x_{12}, 1) := C_{12}(x_1, \dots, x_{12}).$$

This produces a **partial monochromatic line-avoidant coloring** of $[3]^{13}$.

Completing the Construction via SAT

We have colored one slice of $[3]^{13}$.

We encode these constraints as a SAT instance and use a solver to search for a valid completion.

Result: After 360 CPU-days, we identify 9 full colorings of $[3]^{13}$ with no monochromatic combinatorial lines.

Thus,

$$HJ(3;3) \geq 14.$$

Hales–Jewett Variants

Geometric Lines

A *geometric line* is defined by a geometric variable word,

$$w^* \in ([k] \cup \{\star, \diamond\})^n,$$

containing at least one $\star$ and at least one $\diamond$.

For each $a \in [k]$, define a point $w^*(a)$ by:

$$w^*(a)_i = \begin{cases} a & \text{if } w_i^* = \star, \\ k + 1 - a & \text{if } w_i^* = \diamond, \\ w_i^* & \text{otherwise.} \end{cases}$$

Some coordinates **increase** while others **decrease.**

Example of a Geometric Line

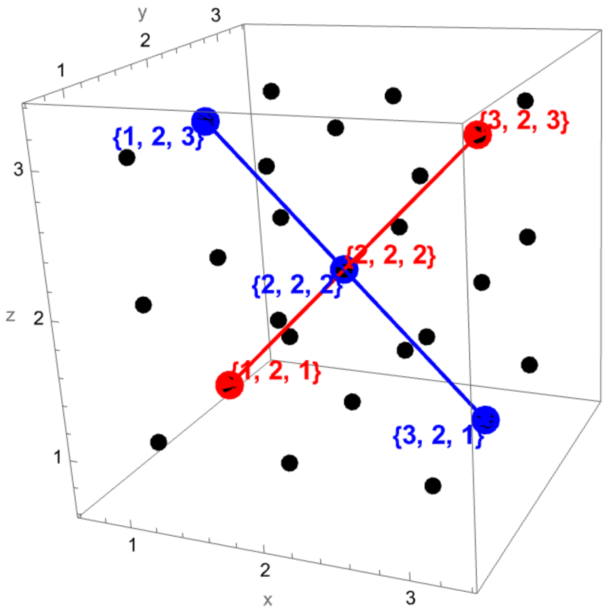
Let $k = 3$, $n = 3$, and consider the geometric variable word

$$w^* = (\star, 2, \diamond).$$

For each $a \in [3] = \{1, 2, 3\}$, we construct points $w^*(a)$:

$$w^*(1) = (1, 2, 3),$$
$$w^*(2) = (2, 2, 2),$$
$$w^*(3) = (3, 2, 1).$$

Visualizing Geometric and Combinatorial Lines



Geometric Variants

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- $G(k; r)$: Hales–Jewett number for geometric lines
- $HJ^*(k; r)$: Hales–Jewett number for combinatorial or geometric lines

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Results:

$\boldsymbol{k}$	$G(k; 2)$	$HJ^*(k; 2)$
2	3	2
3	**5**	**3**
4	$\geq$ **8**	$\geq$ **7**

Off-diagonal Hales–Jewett Numbers

A **partial *m*-line** is a subset of a line consisting of $m \leq k$ consecutive points.

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Off-diagonal Hales–Jewett Numbers

Let $m \leq k$. Then $HJ(k, m; 2)$ is the minimal n such that every 2-coloring of $[k]^n$ admits:

- a red combinatorial k-line, or
- a blue combinatorial m-line.

Given $j \leq m \leq k$, $HJ(k, m, j; 3)$ guarantees at least one of:

- red: k-line, blue: m-line, green: j-line.

Analogous extensions apply to $G(k; r)$ and $HJ^*(k; r)$.

Off-diagonal Results

k	$HJ(k,2;2)$	$G(k,2;2)$	$HJ^*(k,2;2)$
2	2	**3**	**2**
3	3	**4**	**2**
4	4	**4**	**3**
5	5	**6**	**4**
6	**6**	**6**	**4**
7	**7**	$\geq$ **8**	**5**
8	$\geq$ **7**	**7**	**4**

3-Color Results:

$$\boldsymbol{HJ(3,2,2;3) = 5, \quad HJ(3,3,2;3) = 9}$$

A Conjecture

Conjecture

For all $k \geq 2$,

$$HJ(k, 2; 2) = k.$$

Theorem (Conlon, F-, Robertson)

$$HJ(k, 2; 2) \leq k \quad \text{and} \quad HJ(k, 2; 2) \geq k - t$$

where $k - t$ is the largest prime $\leq k$.

The Remaining Gap

If p is prime, then

$$HJ(p, 2; 2) = p.$$

To prove the conjecture, it suffices to show:

$$HJ(k, 2; 2) \geq k \quad \text{for composite } k.$$

Future Directions

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- The resolution of the off-diagonal Hales–Jewett conjecture
- The improvement of SAT solver performance on Hales–Jewett instances
- The application of SAT to related combinatorial invariants

Acknowledgements and References

Acknowledgments

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References

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Appendix A: $HJ(p, 2) \geq p$

Consider a 2-coloring of $[p]^{p-1}$ and color a point x blue iff

$$\sum_{i=1}^{p-1} x_i \equiv 0 \pmod{p}.$$

Along a line the sum becomes

$$a + td \pmod{p},$$

and we obtain a blue point exactly when $a + td \equiv 0 \pmod{p}$.
Since p is prime, d is invertible mod p, so a solution

$$t \equiv -ad^{-1} \pmod{p}$$

always exists.